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Numerical methods and their applications in fisheries Science

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Dedication

To my dearest parents, my father Mouloud and my mother Siham,
Thank you for teaching me the true meaning of perseverance throughout my studies. Your sacrifices, wise advice, and unwavering encouragement have been the foundation of my journey. Nothing in this world compares to the effort you made, day and night, for my education and well-being. This work is the fruit of your countless sacrifices, and I owe it all to you.

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Numerical methods and their applications in fisheries science

Abstract

Mathematical modeling plays a crucial role in understanding and predicting the behavior of complex real-world systems. Differential equations, particularly fractional-order models, provide a powerful framework for capturing the dynamics of systems where memory effects and non-local behaviors are significant. This thesis extends a classical predator-prey fishery model by incorporating the Caputo fractional derivative to account for ecological memory effects, which are essential for realistic predictions in marine population dynamics. The model explores the interactions between prey and predator species distributed across two zones: an open-access fishing area and a protected refuge. The work utilizes the Atangana-Toufik numerical method, a novel approach for solving fractional differential equations, to simulate and analyze the system's behavior. The results highlight the influence of fractional-order dynamics on population stability, harvesting strategies, and the resilience of marine ecosystems. This study provides insights into the role of fractional calculus in ecological modeling and offers practical guidance for sustainable fishery management, emphasizing the importance of considering long-term dependencies in decision-making processes.

Keywords : Atangana-Toufik method, Fractional differential equations, Predator-prey Models, Sustainable Fishery Management.

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Méthodes numériques et leurs applications en science des pêches

RÉSUMÉ

La modélisation mathématique joue un rôle crucial dans la compréhension et la prédiction du comportement des systèmes complexes du monde réel. Les équations différentielles, en particulier les modèles d'ordre fractionnaire, offrent un cadre puissant pour capturer la dynamique des systèmes où les effets de mémoire et les comportements non locaux sont significatifs. Cette thèse étend un modèle classique de prédateur-proie en incorporant la dérivée fractionnaire de Caputo pour tenir compte des effets de mémoire écologiques, essentiels pour des prévisions réalistes des dynamiques des populations marines. Le modèle explore les interactions entre les espèces proies et prédateurs réparties sur deux zones : une zone de pêche libre et une zone protégée. Ce travail utilise la méthode numérique d'Atangana-Toufik, une approche innovante pour résoudre les équations différentielles fractionnaires, afin de simuler et d'analyser le comportement du système. Les résultats mettent en évidence l'influence de la dynamique d'ordre fractionnaire sur la stabilité des populations, les stratégies de pêche et la résilience des écosystèmes marins. Cette étude offre des éclairages sur le rôle du calcul fractionnaire dans la modélisation écologique et fournit des orientations pratiques pour une gestion durable des pêcheries, en soulignant l'importance de considérer les dépendances à long terme dans le processus de prise de décision.

MOTS CLÉS : Méthode d'Atangana-Toufik, Équations differentielles fractionnaires, Modèles prédateur-proie, Gestion durable des pêcheries.

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Introduction

1

Introduction

Understanding how systems evolve over time whether in physics, biology, or economics requires a mathematical language capable of describing change. Differential equations provide this language. They form the foundation for expressing the laws that govern dynamic systems by relating a function to its rates of change. Whether modeling the motion of planets, the spread of disease, or the interaction between predator and prey, differential equations serve as a bridge between short-term behavior and long-term outcomes. Their value lies not only in representing change but in enabling us to simulate, predict, and interpret complex real-world phenomena . Through the modeling process, one translates scientific or physical systems into mathematical form, explores the properties of the resulting equations, and interprets these findings back into the original context to inform understanding or guide decision-making [1, 2, 3, 4, 5, 6, 7, 8].

Mathematical modeling plays a crucial role in analyzing and predicting the behavior of real-world systems. From ecological dynamics to engineering processes, models constructed through differential equations help researchers describe the essential mechanisms driving natural and human-influenced phenomena. However, the utility of these models lies not merely in their formulation, but in their solution. To gain insights, test hypotheses, or make predictions, we must be able to solve the equations governing these systems.

There are fundamentally two approaches to solving differential equations: analytical and numerical. Analytical methods aim to derive exact solutions using closed-form expressions. While powerful and insightful, these methods often fall short when dealing with nonlinear or complex systems, particularly those involving fractional or partial differential equations [? 9, 10, 11, 12, 13, 14, 15, 16]. Such limitations have driven the increasing reliance on numerical techniques designed to approximate solutions with a predictable level of precision.

Over recent decades, a broad range of numerical schemes have been developed to address these challenges. Newer approaches have been shown to offer greater precision and efficiency. In particular, Atangana and İğret Araz introduced a novel class of numerical algorithms based on Newton polynomial interpolation [17], which has demonstrated improved accuracy in solving nonlinear and fractional-order differential equations.

Covering over 70% of the Earth's surface, the world's oceans play a crucial role in regulating climate, shaping weather patterns, and supporting an extraordinary range of biodiversity. Within this vast marine ecosystem, fisheries represent one of the most vital and economically significant human interactions with oceanic life. Fish not only serve as a critical source of protein for billions of people worldwide but also contribute significantly to the global economy through trade, employment, and food security.

However, unsustainable fishing practices have placed immense pressure on marine resources. According to global estimates, approximately 70% of fish stocks are overexploited or harvested beyond biologically sustainable levels. This overfishing, often driven by both legal and illegal exploitation, environmental degradation, and geopolitical competition over marine zones, has led to the decline of key species , the loss of biodiversity , and the destabilization of ecosystems . Climate change and pollution have further compounded these challenges, altering ocean temperatures, disrupting habitats, and weakening the resilience of marine populations.

In response to these pressing issues, fisheries science has increasingly turned to mathematical and computational modeling to better understand and manage marine populations.Models enable researchers to estimate the maximum amount of resources that can be harvested without depleting the population, predict the long-term consequences of harvesting strategies, and assess the effectiveness of interventions such as marine protected areas or no-fishing zones. Among the most widely studied frameworks are predator-prey systems , which explore how the interaction between different species, alongside human harvesting efforts, influences population dynamics.

Recent research has introduced fractional-order differential equations into fishery modeling to better capture the memory effects inherent in ecological systems. Integer-order models often fail to account for these long-term dependencies and feedback mechanisms, which are essential to understanding how ecosystems respond to both natural fluctuations and human interference.

Fractional calculus , particularly through derivatives like the Caputo-Fabrizio operator , provides a more flexible and realistic framework by incorporating non-local, history-dependent behavior into the system dynamics [18, 19, 20, 21, 22, 23, 24, 25, 26].

In this work, we extend a classical predator-prey fishery model by incorporating the Caputo fractional derivative to study the population dynamics of prey and predator species distributed across two zones: an open-access fishing area and a protected refuge where neither harvesting nor predation occurs. This spatial division reflects real-world conservation strategies and introduces important ecological considerations such as migration, refuge effects, and spatial heterogeneity. Our goal is to investigate how fractional-order modeling alters the behavior of the system, improves predictions, and offers deeper insight into sustainable fishery management.

To solve this complex system, which lacks an explicit analytical solution, we implement a numerical approach based on the Atangana-Toufik method [27]. This scheme, rooted in the fundamental theorem of fractional calculus and utilizing a two-step Lagrange polynomial interpolation, is specifically designed for fractional differential equations with non-local and non-singular kernels such as the Caputo-Fabrizio derivative. Unlike traditional predictor-corrector methods, it offers rapid convergence and high accuracy in solving both linear and nonlinear problems. We apply this approach to simulate the dynamics of the proposed fishery model and analyze the impact of varying the fractional order on system behavior. Furthermore, we investigate the stability of equilibrium points and compare the outcomes of fractional-order and classical models, highlighting the significant influence of memory effects on population dynamics under harvesting pressure and environmental variability.

This investigation not only contributes to the theoretical understanding of fractional differential equations in ecological modeling but also provides practical insights into the design of more effective and resilient fishery policies.

This introduction is followed by five additional sections. In Section 2, we provide the necessary mathematical preliminaries, including definitions and properties of the fractional derivatives used in this study. Section 3 presents an overview of previous and related works in the field, and introduces our proposed fractional-order fishery model. In Section 4, we apply the Atangana-Toufik numerical method to the model, including a convergence analysis of the scheme. Section 5 discusses the numerical simulations and results, highlighting key insights into the system's behavior under different conditions. Finally, Section 6 concludes the thesis, summarizing the main findings and suggesting directions for future research.

Preliminaries

2

Preliminaries

LAGRANGE INTERPOLATING POLYNOMIAL

The Lagrange interpolating polynomial is a polynomial $P(x)$ of degree at most $n-1$, which passes through n distinct data points.

$$(x_1, y_1 = f(x_1)), \quad (x_2, y_2 = f(x_2)), \quad \ldots, \quad (x_n, y_n = f(x_n)).$$

It is defined as:

$$P(x) = \sum_{j=1}^{n} P_j(x), \tag{2.1}$$

where

$$P_j(x) = y_j \prod_{\substack{k=1 \\ k\neq j}}^{n} \frac{x - x_k}{x_j - x_k}. \tag{2.2}$$

Written explicitly, the polynomial becomes:

$$P(x) = \frac{(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)} y_1 + \cdots + \frac{(x-x_1)(x-x_2)\cdots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})} y_n. \tag{2.3}$$

EULER'S GAMMA FUNCTION

Definition 2.1. The Gamma function is defined as [28]:

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} \, \mathrm{d}t, \tag{2.4}$$

for all $x > 0$. This integral defines a function $\Gamma : (0, \infty) \to \mathbb{R}$, known as the *Gamma function of Euler*, or equivalently, Euler's integral of the second kind.

FUNDAMENTAL PROPERTIES

1. Value at 1:

$$\Gamma(1) = 1.$$

2. Recursive Relation: For any $x > 0$,

$$\Gamma(x+1) = x\Gamma(x).$$

3. Relation to the Factorial Function: For every non-negative integer n,

$$\Gamma(n+1) = n!.$$

POWER LAW FUNCTION

Definition 2.2. The expression

$$\mathfrak{f}(x) := x^{\mathfrak{a}} \tag{2.5}$$

is known as a *power-law function*, where $\mathfrak{a}$ denotes the exponent that characterizes the power.

RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL

Definition 2.3. The *Riemann–Liouville fractional integral* of a function $\mathfrak{f}(x)$ of order $\alpha > 0$ is defined as [29]:

$$ {}^{RL}_{\ a}\mathcal{I}^{\alpha}_{x}\,\mathfrak{f}(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\mathfrak{f}(\xi)}{(x-\xi)^{1-\alpha}}\,\mathrm{d}\xi, \tag{2.6}$$

for all $x \in [a, b]$, where the interval satisfies $-\infty \leq a < b \leq +\infty$.

We now list several useful properties of the Riemann-Liouville fractional integral, stated without proof:

1. Semigroup Property: For any $\alpha, \beta > 0$ and $x \in [a, b]$, the composition of two Riemann–

Liouville integrals satisfies:

$$\left({}_{a}^{RL}\mathcal{I}_{x}^{\alpha}\, {}_{a}^{RL}\mathcal{I}_{x}^{\beta}\, \mathfrak{f}\right)(x) = {}_{a}^{RL}\mathcal{I}_{x}^{\alpha+\beta}\, \mathfrak{f}(x).$$

2. Action on Power Functions (Case 1): If $\mathfrak{f}(x) = x^{\beta}$, where $\beta > -1$, then the fractional integral evaluates to:

$$ {}_{a}^{RL}\mathcal{I}_{x}^{\alpha}\, x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}\, x^{\alpha+\beta}.$$

3. Action on Shifted Power Functions (Case 2): If $\mathfrak{f}(x) = (x-a)^{\gamma}$, where $\gamma > 0$, then:

$$ {}_{a}^{RL}\mathcal{I}_{x}^{\alpha}\,(x-a)^{\gamma} = \frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\,(x-a)^{\gamma+\alpha-1}.$$

4. Linearity: The Riemann–Liouville fractional integral is linear. For constants C_1, C_2 and functions $\mathfrak{f}, \mathfrak{g}$, we have:

$$ {}_{a}^{RL}\mathcal{I}_{x}^{\alpha}\,(C_1\mathfrak{f} + C_2\mathfrak{g})(x) = C_1\, {}_{a}^{RL}\mathcal{I}_{x}^{\alpha}\, \mathfrak{f}(x) + C_2\, {}_{a}^{RL}\mathcal{I}_{x}^{\alpha}\, \mathfrak{g}(x).$$

Caputo Fractional Derivative

Definition 2.4. Let $\mathfrak{f} \in C_{-1}^{m}[a,b]$. The *Caputo fractional derivative* of order α is defined as [29]:

$$ {}_{a}^{C}\mathcal{D}_{x}^{\alpha}\mathfrak{f}(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_{a}^{x} \frac{\mathfrak{f}^{(m)}(\xi)}{(x-\xi)^{1+\alpha-m}}\,\mathrm{d}\xi, & \text{if } m-1 < \alpha < m, \quad m \in \mathbb{N}, \\ \mathfrak{f}^{(m)}(x), & \text{if } \alpha = m, \quad m \in \mathbb{N}. \end{cases}$$

Below are some fundamental properties of the Caputo fractional derivative that are particularly useful in applications (see [29]):

1. Linearity: For constants C_1, C_2 and functions $\mathfrak{f}, \mathfrak{g}$,

$$ {}_{a}^{C}\mathcal{D}_{x}^{\alpha} \left(C_1\mathfrak{f} + C_2\mathfrak{g}\right)(x) = C_1\, {}_{a}^{C}\mathcal{D}_{x}^{\alpha}\mathfrak{f}(x) + C_2\, {}_{a}^{C}\mathcal{D}_{x}^{\alpha}\mathfrak{g}(x). $$

2. Compositional Property: If $\alpha \in (m-1, m)$, $n \in \mathbb{N}$, then

$$ \left({}_{a}^{C}\mathcal{D}_{x}^{\alpha}\, {}_{a}^{C}\mathcal{D}_{x}^{n}\mathfrak{f}\right)(x) = {}_{a}^{C}\mathcal{D}_{x}^{\alpha+n}\mathfrak{f}(x). $$

3. Inverse Relationship with Riemann-Liouville Integral: For $\alpha > 0$, $m = \lceil \alpha \rceil + 1$, and $\mathfrak{f} \in AC[a, b]$,

$$ \left({}_{a}^{RL}\mathcal{I}_{x}^{\alpha}\, {}_{a}^{C}\mathcal{D}_{x}^{n}\mathfrak{f}\right)(x) = \mathfrak{f}(x) - \sum_{k=0}^{m-1} \frac{\mathfrak{f}^{(k)}(a)}{k!}(x-a)^{k}. $$

Related works

3

Related Works

In recent years, the dynamics of plankton populations, particularly the interactions between phytoplankton and zooplankton, have attracted significant attention. Several mathematical models have been developed to explore these interactions, focusing on various aspects such as nutrient cycling, toxin production, and the effects of environmental factors on population dynamics.

Das and Ray [30] examined the impact of delay on nutrient cycling in phytoplankton-zooplankton interactions within estuarine systems. They highlighted the importance of temporal delays in nutrient availability and their potential effects on the stability of the ecosystem. Saha and Bandyopadhyay [31] extended this work by considering toxin-producing phytoplankton-zooplankton models, where toxin release by phytoplankton follows a discrete time variation. Their model aimed to investigate the influence of toxin dynamics on the survival and growth of both phytoplankton and zooplankton populations.

In further studies, various models have been proposed to investigate the effects of toxic substances on phytoplankton and zooplankton populations. For example, Chattopadhyay et al. [32] developed models involving nutrient-plankton interactions, where the presence of toxins inhibited the growth rates of either phytoplankton, zooplankton, or both trophic levels. Similarly, Roy [33] constructed a model to describe the interaction between toxic and non-toxic phytoplankton under a single nutrient regime, examining the consequences of toxic phytoplankton on zooplankton populations.

Despite these advances, the understanding of the effects of harvest on plankton populations remains limited. Recent studies emphasize the perturbations caused by human activities, particularly overfishing, which can severely impact the stability of marine ecosystems. For instance, the rapid technological advances and growing human populations in the past century have contributed to a significant depletion of global fish stocks. As a result, sustainable management of renewable resources, including plankton populations, has become crucial. Saha and Bandyopadhyay [31] and Chattopadhyay et al. [32] have contributed to the understanding of this issue by considering optimal harvesting strategies within the context of plankton-zooplankton models.

In their work, Saha and Bandyopadhyay [31] introduced a toxic-phytoplankton-zooplankton system, where the authors focused on nutrient cycling and toxin production as key drivers of ecosystem dynamics. This model, which has been adapted by subsequent authors, provides a foundation for investigating the effects of harvesting on both phytoplankton and zooplankton populations. Their differential equations model considers factors such as phytoplankton growth rates, zooplankton grazing rates, and toxin production, all while incorporating the effects of harvesting through appropriate catchability coefficients (c_1 and c_2).

The two systems of differential equations presented in their models are as follows [31]:

$$\begin{aligned} \frac{dP}{dt} &= rP\left(1 - \frac{P}{k}\right) - \frac{\beta PZ}{\kappa + P}, \\ \frac{dZ}{dt} &= \frac{\beta_1 PZ}{\kappa + P} - dZ - \frac{\rho PZ}{\kappa + P}. \end{aligned} \tag{3.1}$$

Where:

- P represents the density of toxin-producing phytoplankton (TPP) population.
- Z represents the density of zooplankton population.
- r is the intrinsic growth rate of phytoplankton.
- k is the carrying capacity of the environment for the phytoplankton population.
- β is the maximum uptake rate for zooplankton, while β_1 is the biomass conversion rate.
- κ is the half-saturation constant for the Holling type II functional response.
- d is the natural death rate of zooplankton.
- ρ is the rate of toxic substance production by phytoplankton.

The assumptions of the model include:

1. The populations of both phytoplankton and zooplankton are subject to initial non-negative conditions, $P(0) = P_0 \geq 0$ and $Z(0) = Z_0 \geq 0$.
2. The harvesting effort E affects both phytoplankton and zooplankton populations, with distinct coefficients c_1 and c_2 for each species.

In addition to the above system, the authors extended their model to include the effects of harvesting on both phytoplankton and zooplankton populations. The extended system is as follows [34]:

$$\begin{aligned} \frac{dP}{dt} &= rP\left(1 - \frac{P}{k}\right) - \frac{\beta PZ}{\kappa + P} - c_1 EP, \\ \frac{dZ}{dt} &= \frac{\beta_1 PZ}{\kappa + P} - dZ - \frac{\rho PZ}{\kappa + P} - c_2 EZ. \end{aligned} \tag{3.2}$$

Where:

- c_1 and c_2 are the catchability coefficients for phytoplankton and zooplankton, respectively.
- E is the harvesting effort.

This model incorporates the effects of both natural population dynamics and anthropogenic activities (such as harvesting), providing a framework to study the sustainable management of plankton ecosystems.

Marine reserves have become an increasingly viable solution for mitigating the overexploitation of marine species. The paper by Lv et al. [18] expands on the earlier plankton-zooplankton model by introducing a predator-prey model with Holling Type II functional response. In this framework, the authors incorporate the dynamics of migration between two patches, where one patch serves as a reserve area (protected from fishing and predation) and the other remains an open-access fishery zone. The prey species migrate between the two patches, and the predator species primarily consume prey in the unreserved area.

The system of differential equations introduced in this extended model is as follows:

$$\begin{cases} \frac{dx}{dt} &= rx\left(1-\frac{x}{K}\right)-\sigma_1 x+\sigma_2 y-\frac{\mu_1 xz}{\kappa+x}-q_1 Ex, \\ \frac{dy}{dt} &= sy\left(1-\frac{y}{L}\right)+\sigma_1 x-\sigma_2 y, \\ \frac{dz}{dt} &= \frac{\mu_2 xz}{\kappa+x}-dz-q_2 Ez. \end{cases}$$

Where:

- $x(t)$ and $y(t)$ are the biomass densities of the prey species in the unreserved and reserved areas, respectively.
- $z(t)$ is the biomass density of the predator species.
- r and s are the intrinsic growth rates of the prey species in the unreserved and reserved patches.
- K and L are the carrying capacities of the prey species in the unreserved and reserved areas.
- σ_1 and σ_2 are the migration rates between the unreserved and reserved areas.
- q_1 and q_2 are the catchability coefficients of the prey and predator species.
- E is the harvesting effort.
- μ_1 is the maximum uptake rate for the functional response.
- κ is the half-saturation constant for a Holling Type II functional response.

- μ_2 is the biomass conversion rate.
- d is the natural death rate for predator species.

This extension models both migration and functional responses, providing a more comprehensive framework for studying the impact of marine reserves, migration dynamics, and harvesting on prey and predator species. The paper discusses the existence and stability of equilibrium points, and through numerical simulations, it concludes that the existence of marine reserves significantly contributes to the sustainability of the predator-prey system, benefiting both the species within the reserve and adjacent unreserved areas.

The paper by Mansal and Sene [35] further develops the predator-prey model by incorporating fractional-order derivatives, a natural extension to classical differential equations that introduces memory effects in the model. Building on the previous models, the authors introduce the concept of fractional-order derivatives using the Caputo-Fabrizio fractional derivative. This modification allows the model to account for historical influences on the species' population dynamics, offering a more accurate representation of biological systems where memory effects are significant.

The fractional-order differential equations for the system are given by:

$$\begin{cases} D_t^{\alpha,CF}x &= rx\left(1-\frac{x}{K}\right)-\sigma_1 x+\sigma_2 y-\frac{\mu_1 xz}{\kappa+x}-q_1 Ex, \\ D_t^{\alpha,CF}y &= sy\left(1-\frac{y}{L}\right)+\sigma_1 x-\sigma_2 y, \\ D_t^{\alpha,CF}z &= \frac{\mu_2 xz}{\kappa+x}-dz-q_2 Ez. \end{cases}$$

Where:

- $D_t^{\alpha,CF}$ represents the Caputo-Fabrizio fractional derivative.

- $r, s, \sigma_1, \sigma_2, \mu_1$, and μ_2 are the parameters as previously described.
- The fractional derivatives introduce a memory effect, which modifies how the population dynamics are influenced by past states.

This formulation offers a more generalized approach to modeling population dynamics, incorporating the effects of memory into the system's behavior. Mansal and Sene [35] analyzes the existence and stability of solutions for the fractional-order system and explores how this modification impacts the long-term sustainability of the predator-prey ecosystem under different harvesting conditions.

The model we are working with is based on the fractional-order differential equations presented above. This system will be solved using the Atangana-Toufik method, a powerful approach for handling fractional differential equations. The Atangana-Toufik method is well-suited for solving systems of equations with fractional derivatives and can provide a better approximation of real-world biological systems that exhibit memory effects.

The initial system of fractional-order differential equations, governed by the Caputo derivative, is as follows:

$$\begin{cases} {}^{C}D_t^{\alpha}x(t) = rx\left(1-\frac{x}{k}\right) - \sigma_1 x + \sigma_2 y - \frac{\mu_1 xz}{\kappa + x} - q_1 Ex, \\ {}^{C}D_t^{\alpha}y(t) = sy\left(1-\frac{y}{L}\right) + \sigma_1 x - \sigma_2 y, \\ {}^{C}D_t^{\alpha}z(t) = \frac{\mu_2 xz}{\kappa + x} - dz - q_2 Ez, \end{cases} \qquad 0 < \alpha \leq 1,\ t \in [0, T]. \tag{3.3}$$

This system is defined on the time interval $[0, T]$, where T is the final time for which the solution will be computed. The Atangana-Toufik method will be applied to solve this system of equations, providing a numerical solution that incorporates the fractional-order derivatives and memory effects. The solution will give insight into the population dynamics of prey and predator species, taking into account the harvesting efforts and the impact of the fractional calculus on the system's behavior.

AtanganaToufik Method

4

Atangana-Toufik Method

In this chapter, we will apply the Atangana-Toufik method to solve a fractional-order predator-prey model that incorporates the Caputo fractional derivative. The model, based on the fractional-order differential equations outlined in the previous section, describes the dynamics of three interacting species-prey, predator, and a third species-while accounting for memory effects that are inherent in real-world biological systems. The introduction of fractional-order derivatives allows for the inclusion of historical influences on the system's population dynamics, making the model more accurate and representative of ecological processes that exhibit memory effects.

We begin with the initial system of fractional-order differential equations, as expressed by:

$$\begin{cases} {}^{C}D_t^{\alpha}x(t) = \frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \sigma_1 x + \sigma_2 y - \frac{\mu_1 xz}{\kappa + x} - q_1 Ex = F(x, y, z, t), & \\ {}^{C}D_t^{\alpha}y(t) = \frac{dy}{dt} = sy\left(1 - \frac{y}{L}\right) + \sigma_1 x - \sigma_2 y = H(x, y, z, t), & t \in [0, T]. \\ {}^{C}D_t^{\alpha}z(t) = \frac{dz}{dt} = \frac{\mu_2 xz}{\kappa + x} - dz - q_2 Ez = G(x, y, z, t), & \end{cases} \tag{4.1}$$

The next step is to use fractional integrals to express the variables $x(t)$, $y(t)$, and $z(t)$ in terms of their initial values and integrals over time. These are given by:

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} F(x, y, z, \tau)\, d\tau, \tag{4.2}$$

$$y(t) = y(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} H(x, y, z, \tau)\, d\tau, \tag{4.3}$$

$$z(t) = z(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} G(x, y, z, \tau)\, d\tau, \tag{4.4}$$

where $\Gamma(\alpha)$ is the Gamma function and $0 < \alpha < 1$.

At a specific time point t_{n+1}, $n = 0, 1, 2, \ldots$, the fractional-order equations can be reformulated as:

$$\begin{aligned}
x(t_{n+1}) &= x(0) + \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} F(x,y,z,\tau)(t_{n+1}-\tau)^{\alpha-1}\, d\tau \\
&= x(0) + \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} F(x,y,z,\tau)(t_{n+1}-\tau)^{\alpha-1}\, d\tau, \qquad (4.5)\\
y(t_{n+1}) &= y(0) + \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} H(x,y,z,\tau)(t_{n+1}-\tau)^{\alpha-1}\, d\tau \\
&= y(0) + \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} H(x,y,z,\tau)(t_{n+1}-\tau)^{\alpha-1}\, d\tau, \qquad (4.6)\\
z(t_{n+1}) &= z(0) + \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} G(x,y,z,\tau)(t_{n+1}-\tau)^{\alpha-1}\, d\tau \\
&= z(0) + \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} G(x,y,z,\tau)(t_{n+1}-\tau)^{\alpha-1}\, d\tau. \qquad (4.7)
\end{aligned}$$

To approximate the functions $F(x,y,z,\tau)$, $H(x,y,z,\tau)$, and $G(x,y,z,\tau)$ on the interval $[t_k, t_{k+1}]$, we use the two-step Lagrange polynomial interpolation:

$$\begin{aligned}
P_k^x(\tau) &= \frac{\tau - t_{k-1}}{t_k - t_{k-1}} F(x(t_k), y(t_k), z(t_k), t_k) - \frac{\tau - t_k}{t_k - t_{k-1}} F(x(t_{k-1}), y(t_{k-1}), z(t_{k-1}), t_{k-1}) \\
&= \frac{1}{h} F(x_k, y_k, z_k, t_k)(\tau - t_{k-1}) - \frac{1}{h} F(x_{k-1}, y_{k-1}, z_{k-1}, t_{k-1})(\tau - t_k), \qquad (4.8)\\
P_k^y(\tau) &= \frac{\tau - t_{k-1}}{t_k - t_{k-1}} H(x(t_k), y(t_k), z(t_k), t_k) - \frac{\tau - t_k}{t_k - t_{k-1}} H(x(t_{k-1}), y(t_{k-1}), z(t_{k-1}), t_{k-1}) \\
&= \frac{1}{h} H(x_k, y_k, z_k, t_k)(\tau - t_{k-1}) - \frac{1}{h} H(x_{k-1}, y_{k-1}, z_{k-1}, t_{k-1})(\tau - t_k), \qquad (4.9)\\
P_k^z(\tau) &= \frac{\tau - t_{k-1}}{t_k - t_{k-1}} G(x(t_k), y(t_k), z(t_k), t_k) - \frac{\tau - t_k}{t_k - t_{k-1}} G(x(t_{k-1}), y(t_{k-1}), z(t_{k-1}), t_{k-1}) \\
&= \frac{1}{h} G(x_k, y_k, z_k, t_k)(\tau - t_{k-1}) - \frac{1}{h} G(x_{k-1}, y_{k-1}, z_{k-1}, t_{k-1})(\tau - t_k), \qquad (4.10)
\end{aligned}$$

where $h = t_k - t_{k-1}$.

Substituting P_k^x, P_k^y, P_k^z into the fractional integral equations, we get

$$\begin{aligned} x_{n+1} = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \Bigg[& \frac{F(x_k, y_k, z_k, t_k)}{h} \int_{t_k}^{t_{k+1}} (\tau - t_{k-1})(t_{n+1} - \tau)^{\alpha-1} d\tau \\ & - \frac{F(x_{k-1}, y_{k-1}, z_{k-1}, t_{k-1})}{h} \int_{t_k}^{t_{k+1}} (\tau - t_k)(t_{n+1} - \tau)^{\alpha-1} d\tau \Bigg], \end{aligned} \tag{4.11}$$

$$\begin{aligned} y_{n+1} = y_0 + \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \Bigg[& \frac{H(x_k, y_k, z_k, t_k)}{h} \int_{t_k}^{t_{k+1}} (\tau - t_{k-1})(t_{n+1} - \tau)^{\alpha-1} d\tau \\ & - \frac{H(x_{k-1}, y_{k-1}, z_{k-1}, t_{k-1})}{h} \int_{t_k}^{t_{k+1}} (\tau - t_k)(t_{n+1} - \tau)^{\alpha-1} d\tau \Bigg], \end{aligned} \tag{4.12}$$

$$\begin{aligned} z_{n+1} = z_0 + \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \Bigg[& \frac{G(x_k, y_k, z_k, t_k)}{h} \int_{t_k}^{t_{k+1}} (\tau - t_{k-1})(t_{n+1} - \tau)^{\alpha-1} d\tau \\ & - \frac{G(x_{k-1}, y_{k-1}, z_{k-1}, t_{k-1})}{h} \int_{t_k}^{t_{k+1}} (\tau - t_k)(t_{n+1} - \tau)^{\alpha-1} d\tau \Bigg]. \end{aligned} \tag{4.13}$$

Define

$$A_{k,1} = \int_{t_k}^{t_{k+1}} (\tau - t_{k-1})(t_{n+1} - \tau)^{\alpha-1}\, d\tau, \tag{6.1}$$

$$A_{k,2} = \int_{t_k}^{t_{k+1}} (\tau - t_k)(t_{n+1} - \tau)^{\alpha-1}\, d\tau. \tag{6.2}$$

Letting $t_j = jh$, the coefficients are expressed as:

$$A_{k,1} = \frac{h^{\alpha+1}}{\alpha(\alpha+1)} \left[(n+1-k)^{\alpha}(n-k+2+\alpha) - (n-k)^{\alpha}(n-k+2+2\alpha) \right], \tag{7.1}$$

$$A_{k,2} = \frac{h^{\alpha+1}}{\alpha(\alpha+1)} \left[(n+1-k)^{\alpha+1} - (n-k)^{\alpha}(n-k+1+\alpha) \right]. \tag{7.2}$$

The final form of the equations is

$$x_{n+1} = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \left[\frac{F(x_k, y_k, z_k, t_k)}{h} A_{k,1} - \frac{F(x_{k-1}, y_{k-1}, z_{k-1}, t_{k-1})}{h} A_{k,2} \right], \tag{8.1}$$

$$y_{n+1} = y_0 + \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \left[\frac{H(x_k, y_k, z_k, t_k)}{h} A_{k,1} - \frac{H(x_{k-1}, y_{k-1}, z_{k-1}, t_{k-1})}{h} A_{k,2} \right], \tag{8.2}$$

$$z_{n+1} = z_0 + \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \left[\frac{G(x_k, y_k, z_k, t_k)}{h} A_{k,1} - \frac{G(x_{k-1}, y_{k-1}, z_{k-1}, t_{k-1})}{h} A_{k,2} \right]. \tag{8.3}$$

The Atangana–Toufik method gives the numerical solutions as follows:

$$\begin{aligned} x_{n+1} = x_0 &+ \sum_{k=0}^{n} \frac{h}{\alpha\Gamma(\alpha+2)} F(x_k, y_k, z_k, t_k) \left(\frac{(n+1-k)^{\alpha}}{n-k+2+\alpha} - \frac{(n-k)^{\alpha}}{n-k+2+2\alpha} \right) \\ &- \sum_{k=0}^{n} \frac{h}{\alpha\Gamma(\alpha+2)} F(x_{k-1}, y_{k-1}, z_{k-1}, t_{k-1}) \left(\frac{(n+1-k)^{\alpha+1}}{n-k+1+\alpha} - \frac{(n-k)^{\alpha}}{n-k+1+\alpha} \right), \end{aligned} \tag{9.1}$$

$$\begin{aligned} y_{n+1} = y_0 &+ \sum_{k=0}^{n} \frac{h}{\alpha\Gamma(\alpha+2)} H(x_k, y_k, z_k, t_k) \left(\frac{(n+1-k)^{\alpha}}{n-k+2+\alpha} - \frac{(n-k)^{\alpha}}{n-k+2+2\alpha} \right) \\ &- \sum_{k=0}^{n} \frac{h}{\alpha\Gamma(\alpha+2)} H(x_{k-1}, y_{k-1}, z_{k-1}, t_{k-1}) \left(\frac{(n+1-k)^{\alpha+1}}{n-k+1+\alpha} - \frac{(n-k)^{\alpha}}{n-k+1+\alpha} \right), \end{aligned} \tag{9.2}$$

$$\begin{aligned} z_{n+1} = z_0 &+ \sum_{k=0}^{n} \frac{h}{\alpha\Gamma(\alpha+2)} G(x_k, y_k, z_k, t_k) \left(\frac{(n+1-k)^{\alpha}}{n-k+2+\alpha} - \frac{(n-k)^{\alpha}}{n-k+2+2\alpha} \right) \\ &- \sum_{k=0}^{n} \frac{h}{\alpha\Gamma(\alpha+2)} G(x_{k-1}, y_{k-1}, z_{k-1}, t_{k-1}) \left(\frac{(n+1-k)^{\alpha+1}}{n-k+1+\alpha} - \frac{(n-k)^{\alpha}}{n-k+1+\alpha} \right). \end{aligned} \tag{9.3}$$

These equations represent the numerical solutions for the variables $x_{n+1}, y_{n+1}, z_{n+1}$ based on the Atangana–Toufik method involving fractional calculus terms.

Comprehensive Interpretation of the Results

In this section, we compare the results obtained using two different methods: the Atangana-Toufik method and the Hermite method in Mathematica 11.3. We also examine the effects of varying the fractional parameter α on predator-prey dynamics and analyze the impact of the harvesting rate.

We compute the numerical scheme presented in this section by considering the following assumptions related to the values of the parameters: $r = 5, K = 4, \sigma_1 = 1, \sigma_2 = 1, \mu_1 = 0.93, \mu_2 = 0.9, E = 0.8, \kappa = 0.7, s = 1, L = 3, q_1 = 0.1, q_2 = 0.2$, and $d = 0.3$.

We consider the following initial conditions for the entire section: $x(1) = 0.3, y(1) = 0.6, z(1) = 6.5$, and $t(1) = 0$.

For the numerical method, the time step h is set to 0.01, and the interval is $[0, 50]$. All computations were performed using MATLAB R2015a to obtain the results.

4.0.1 Comparison Between Atangana-Toufik and Hermite Methods

Figures 4.1, 4.3, and 4.5 show the solutions for $x(t), y(t)$, and $z(t)$ obtained using the Atangana-Toufik method for $\alpha = 1$. These results are compared with Figures 4.2, 4.4, and 4.6, obtained using the Hermite method in Mathematica 11.3. As seen in the figures, the solutions from both methods are very similar, indicating that both methods produce consistent and reliable results for the predator-prey dynamics. The oscillatory behavior observed in the predator and prey populations is captured well by both methods, confirming the robustness of the model.

4.0.2 RESULTS FOR $\alpha = 0.95$

Figures 4.7, 4.8, and 4.9 represent the solutions for $x(t)$, $y(t)$, and $z(t)$ when $\alpha = 0.95$. As the fractional-order parameter α decreases from 1, the system exhibits oscillatory dynamics, but with more dampened fluctuations compared to the case when $\alpha = 1$. Specifically, the amplitude of the oscillations is smaller, and the populations of prey and predators exhibit a slower response to changes. The prey populations (both in the unreserved and reserved areas) and the predator population show less pronounced oscillations, indicating that the memory effects introduced by the fractional-order derivative are having a dampening effect on the population dynamics. This suggests that the system becomes less sensitive to immediate changes and the populations adjust more gradually over time compared to the case when $\alpha = 1$.

4.0.3 JOINT DYNAMICS OF $x(t), y(t)$, AND $z(t)$ FOR $\alpha = 1$

The system of differential equations we modeled with $\alpha = 1$ (integer-order system) represents a predator-prey interaction with migration dynamics between a reserved area (protected from fishing and predation) and an unreserved area (subject to both). The simulation results for the interval $t \in [0, 50]$ reveal the complex and synchronized dynamics between the prey and predator populations. Initially, the prey biomass in the unreserved area $x(t)$ starts at a low level, while the prey biomass in the reserved area $y(t)$ begins higher, reflecting the protective effects of the reserve. The predator population $z(t)$ starts at a relatively high value compared to both prey populations, indicating that predators have a stable food source at the initial state.

As time progresses, the dynamics of both prey populations, $x(t)$ and $y(t)$, are closely linked, and the oscillations they exhibit occur in a synchronized manner. Specifically, we observe that the peaks of $x(t)$ and $y(t)$ occur almost simultaneously throughout the simulation, suggesting that migration between the two areas (with rates σ_1 and σ_2) is strong and helps replenish the unreserved area with prey from the reserve. This migration leads to an increase in the unreserved prey population $x(t)$ after each peak in the reserve population $y(t)$, and vice versa.

The predator biomass $z(t)$ exhibits a noticeable phase lag compared to the prey populations. After each increase in prey, particularly in $x(t)$, the predator population increases due to the greater food availability, as expected from the Holling Type II functional response. The predator population reaches its peak after the prey peaks, confirming the delayed response of predators to increased prey density. However, after the peak in predator biomass, $z(t)$ begins to decrease slightly, reflecting the top-down control that predators exert on the prey population. This decline in $z(t)$ occurs because the prey population $x(t)$ starts to decrease due to the increased predation pressure, which leads to a reduction in food availability for the predators.

The results highlight the importance of marine reserves in maintaining ecosystem stability. The migration dynamics between the two patches help stabilize both predator and prey populations. The predator-prey dynamics, as observed in the system, are tightly coupled, with migration acting as a key factor in regulating these dynamics. Moreover, the model suggests that harvesting and predation pressure can be mitigated by the presence of a reserve, which helps sustain predator and prey populations over time. The observed oscillatory behavior indicates a system in dynamic equilibrium, where both predator and prey coexist sustainably under the influence of migration and harvesting efforts.

4.0.4 PHASE PLOTS FOR $x(t)$, $y(t)$, AND $z(t)$

Figures 4.14, 4.15, and 4.16 present phase plots of $x(t)$ versus $y(t)$, $x(t)$ versus $z(t)$, and $y(t)$ versus $z(t)$, respectively. The phase plots reveal key insights into the interactions between the prey and predator populations:

- Figure 4.14: The phase plot of $x(t)$ versus $y(t)$ shows the dynamics of the prey populations in the unreserved and reserved areas. The oscillatory nature of the plot indicates that the two populations are strongly interconnected, with periodic fluctuations suggesting migration and interaction between the areas. This behavior reflects the migration of prey from the reserve to the unreserved area, influencing both populations over time.
- Figure 4.15: The phase plot of $x(t)$ versus $z(t)$ shows the predator-prey dynamics between the prey in the unreserved area and the predator species. The predator population increases in response to higher prey biomass but declines as prey availability decreases due to predation. This cycle represents the eedback loop typical of predator-prey interactions, with predators following the availability of prey, but eventually reducing prey numbers through consumption.
- Figure 4.16: The phase plot of $y(t)$ versus $z(t)$ further illustrates the interaction between predator and prey populations in the reserve area. While the predators benefit from the availability of prey in the reserve, once predator numbers increase, they begin to exert negative pressure on the prey biomass. The oscillations suggest a continuous cycle where predators' consumption reduces the prey population, even in a protected area.

The results underscore the importance of fractional-order dynamics in ecological modeling and the significant impact that small changes in parameter values can have on system behavior. The analysis also highlights the crucial role of harvesting efforts in shaping the stability and sustainability of predator-prey systems.

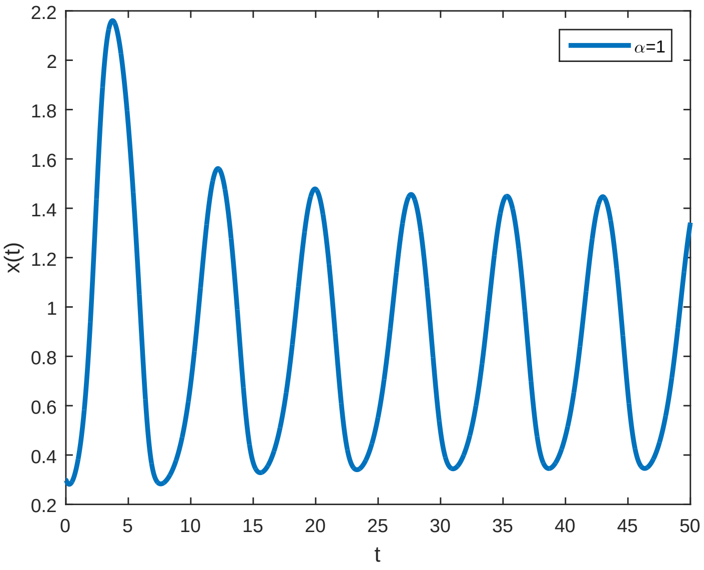

Figure 4.1: Plot of $x(t)$ for $\alpha = 1$ using the Atangana-Toufik method.

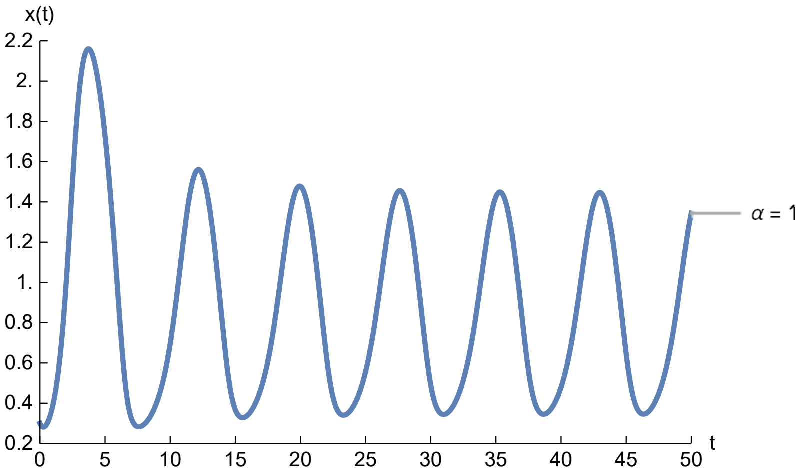

Figure 4.2: Plot of $x(t)$ for $\alpha = 1$ using the Hermite method.

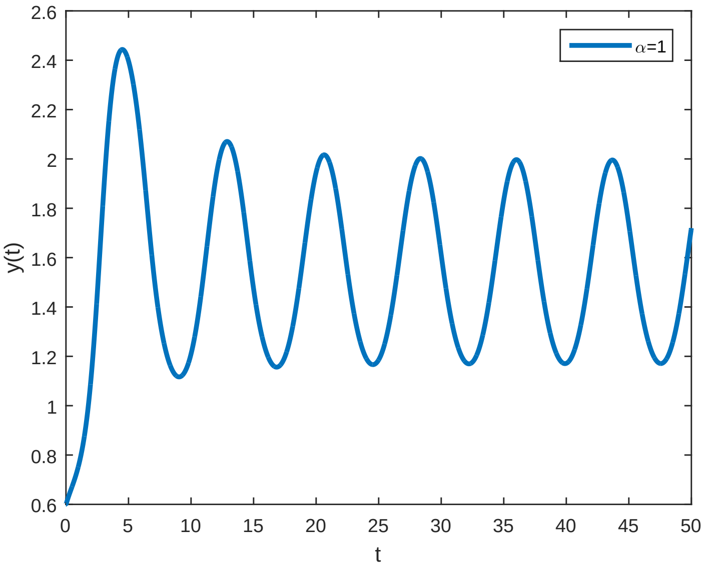

Figure 4.3: Plot of $y(t)$ for $\alpha = 1$ using the Atangana-Toufik method.

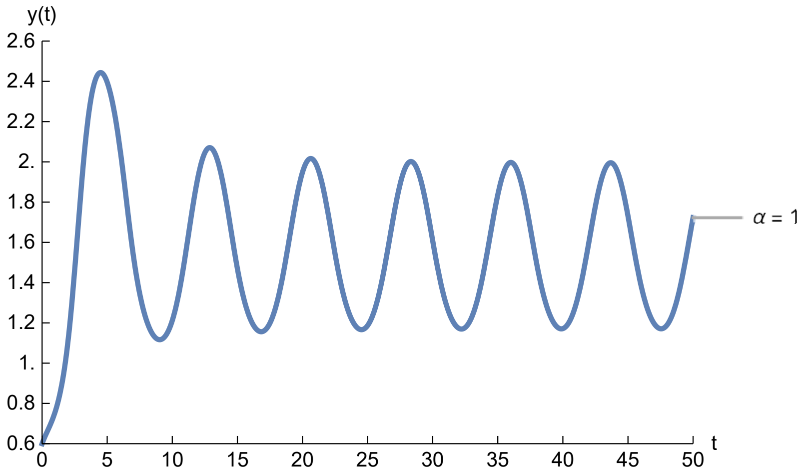

Figure 4.4: Plot of $y(t)$ for $\alpha = 1$ using the Hermite method.

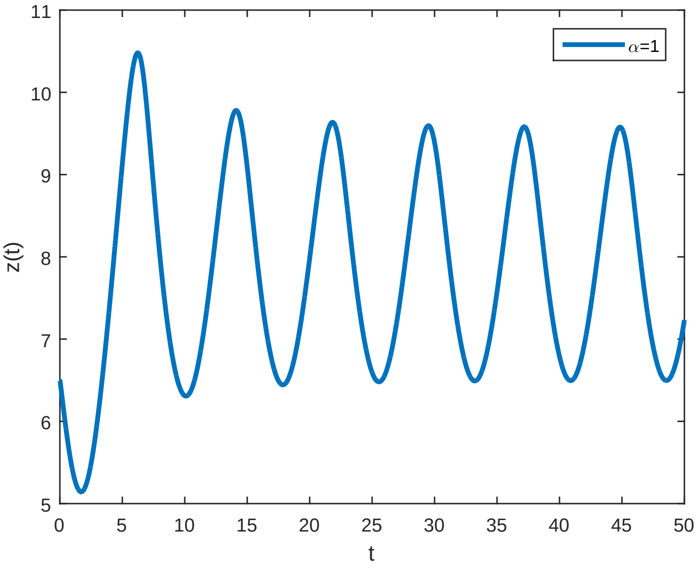

Figure 4.5: Plot of $z(t)$ for $\alpha = 1$ using the Atangana-Toufik method.

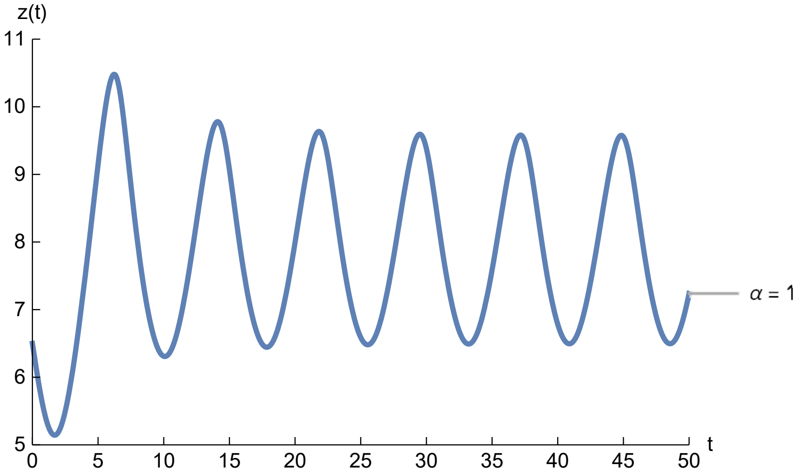

Figure 4.6: Plot of $z(t)$ for $\alpha = 1$ using the Hermite method.

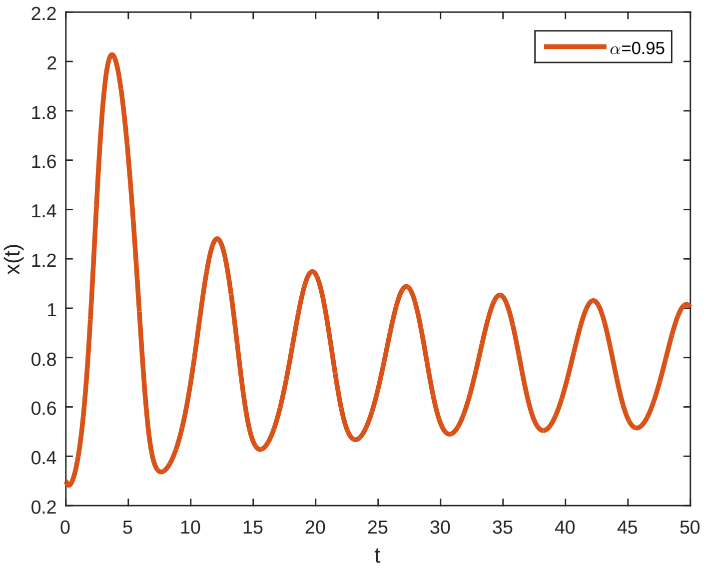

Figure 4.7: Plot of $x(t)$ for $\alpha = 0.95$ using the Atangana-Toufik method.

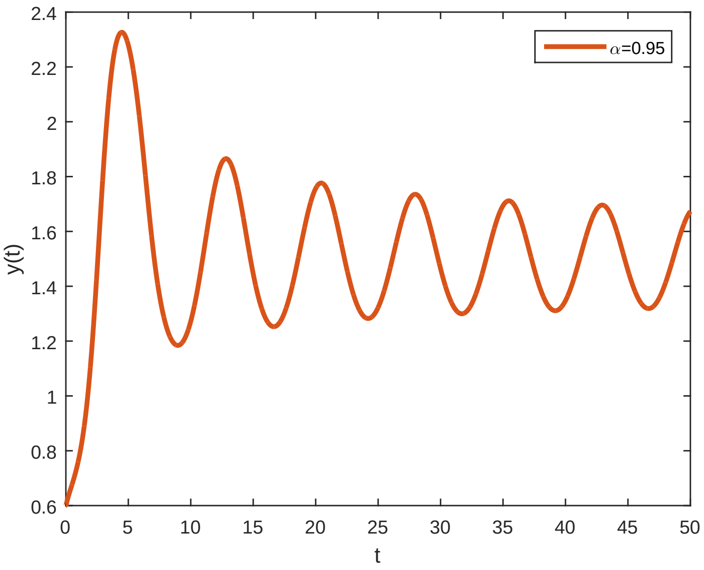

Figure 4.8: Plot of $y(t)$ for $\alpha = 0.95$ using the Atangana-Toufik method.

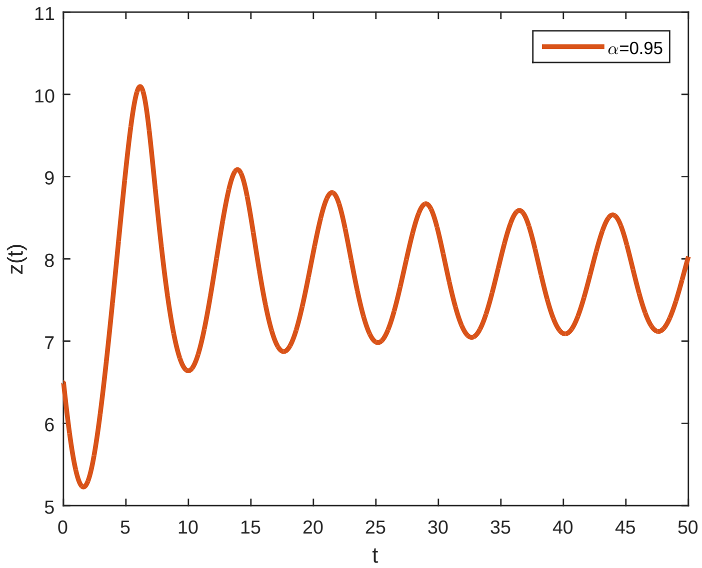

Figure 4.9: Plot of $z(t)$ for $\alpha = 0.95$ using the Atangana-Toufik method.

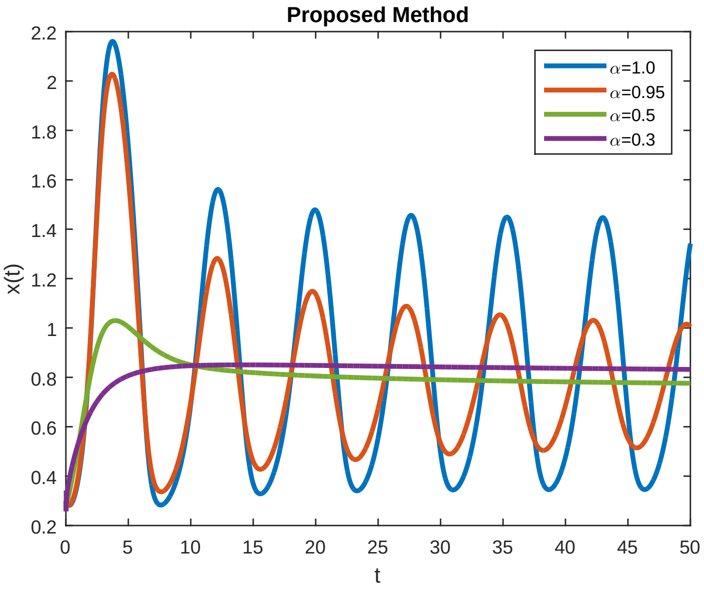

Figure 4.10: Plot of $x(t)$ for different values of α

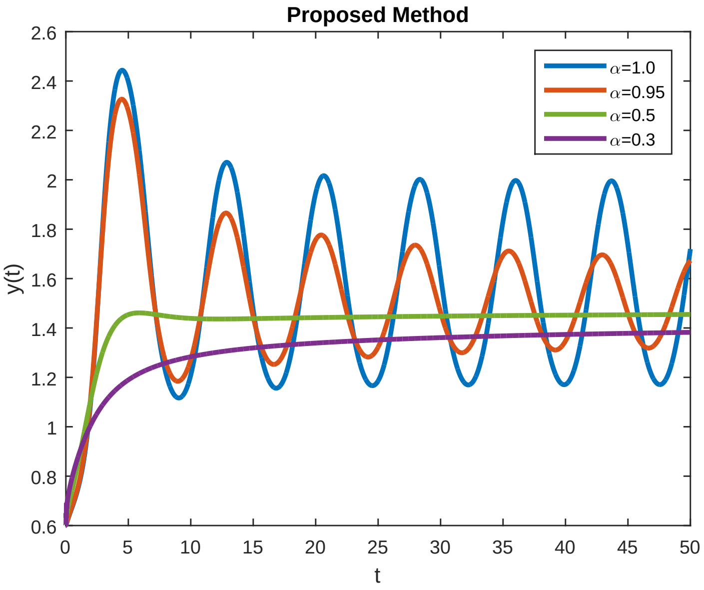

Figure 4.11: Plot of $y(t)$ for different values of α

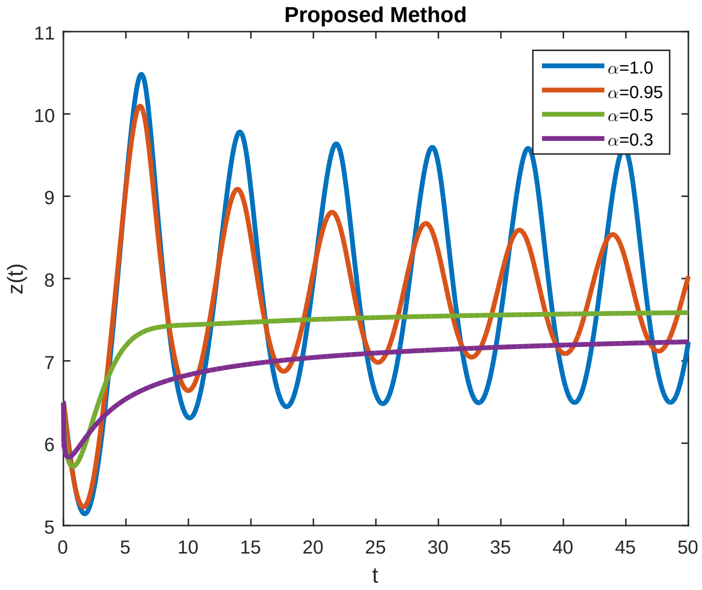

Figure 4.12: Plot of $z(t)$ for different values of α

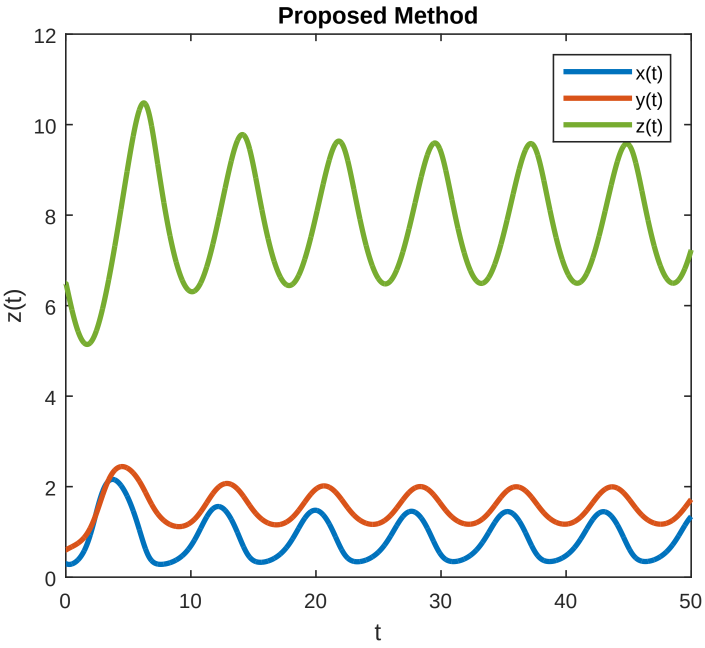

Figure 4.13: Plot of $x(t)$, $y(t)$, and $z(t)$ together for $\alpha = 1$, showing the combined dynamics of the prey and predator populations in the unreserved and reserved areas.

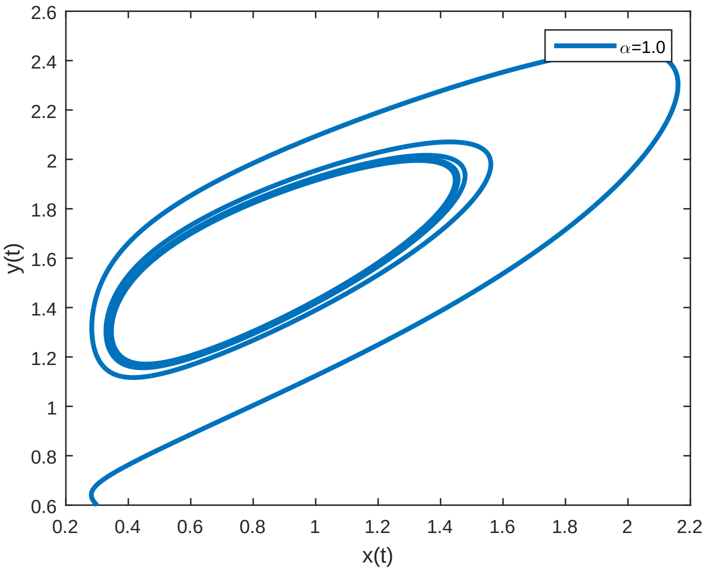

Figure 4.14: Phase plot of $x(t)$ versus $y(t)$ for $\alpha = 1$, showing the dynamics of prey populations in the unreserved and reserved areas.

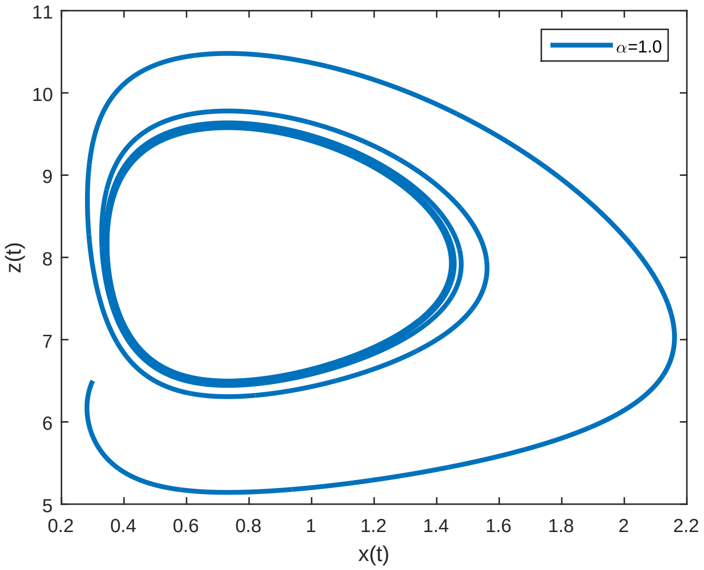

Figure 4.15: Phase plot of $x(t)$ versus $z(t)$ for $\alpha = 1$, illustrating the feedback loop between prey and predator populations.

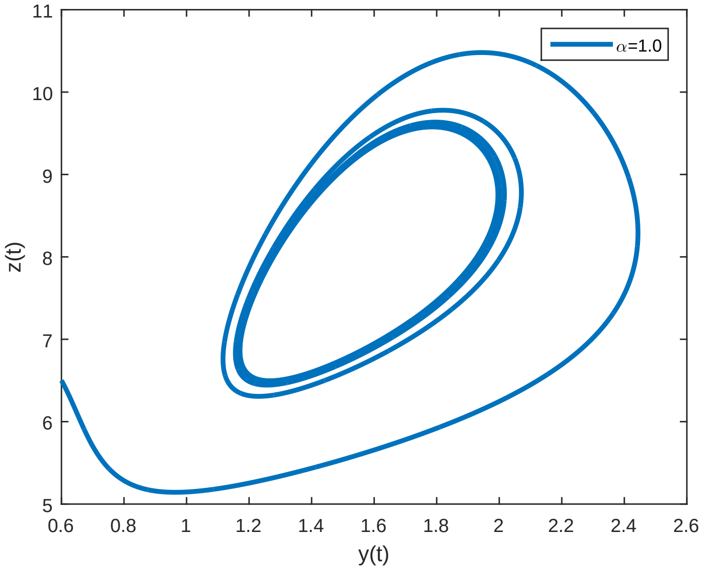

Figure 4.16: Phase plot of $y(t)$ versus $z(t)$ for $\alpha = 1$, depicting the interaction between predator and prey populations in the reserve area.

Conclusion

This thesis presented a novel approach to modeling predator-prey dynamics in marine ecosystems by incorporating fractional-order differential equations using the Caputo fractional derivative. The primary goal was to explore the role of ecological memory effects in shaping the stability and sustainability of fishery systems, specifically in the context of marine reserves and open-access fishing zones.

Through the application of the Atangana-Toufik numerical method, we were able to simulate and analyze the population dynamics of prey and predator species across two spatial zones: one unreserved and subjected to harvesting, and the other a protected reserve. The results revealed that fractional-order modeling, which accounts for memory effects and long-term dependencies in the system, provides a more accurate representation of real-world ecological dynamics than traditional integer-order models.

The phase plot analysis provided valuable insights into the interactions between prey and predator populations in both the unreserved and reserved areas. These interactions highlighted the importance of migration dynamics, where the movement of prey from the reserve to the unreserved area replenishes prey populations and stabilizes the overall system. The predator population exhibited a delayed response to changes in prey biomass, confirming the existence of feedback loops typical of predator-prey interactions.

One of the key findings of this research is the importance of marine reserves in maintaining ecological balance. The reserve area serves as a buffer against overfishing, ensuring that prey populations are protected and can continue to support predator populations over time. This model underscores the significance of incorporating long-term ecological considerations, such as migration, memory effects, and non-local interactions, in the management of marine resources.

In terms of future work, it would be valuable to extend this model by incorporating additional environmental factors such as climate change, pollution, and habitat degradation. Furthermore, exploring the implications of varying harvesting strategies, such as time-based or quota-based regulations, could offer more practical guidance for sustainable fishery management.

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